Van der Waals Excluded Volume Model for Lorentz Contracted Rigid Spheres

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Abstract

Conventional cluster and virial expansions are generalized to momentum dependent interparticle potentials. The model with Lorentz contracted hard core potentials is considered, e.g. as hadron gas model. A Van der Waals-type model with a temperature dependent excluded volume is derived. Lorentz contraction effects at given temperature are stronger for light particles and make their effective excluded volume smaller than that of heavy ones.

Key words: Cluster integrals, hard-core, Van der Waals model, Lorentz contraction.

The Van der Waals (VdW) excluded volume model has been used to describe hadron yields in relativistic nucleus–nucleus collisions (see e.g. [1, 2] and references therein). This model treats the hadrons as hard-core spheres and, therefore, takes into account the hadron repulsion at short distances. In a relativistic situation one should, however, include the Lorentz contraction of the hard core-hadrons. This problem was discussed in the literature (see e.g. Ref. [3, 4]). In this paper the cluster and virial expansions are generalized to velocity dependent inter-particle potentials. This extension is used to construct the VdW model for Lorentz contracted rigid spheres which may be used to simulate hadrons.

The canonical partition function for the gas of N classical (Boltzmann) particles takes the form

$$Z_N(V,T) ~=~ \frac{1}{N!}\int\prod_{i=1}^{N}\left[\frac{g~d\mathbf{r}_i d\mathbf{k}_i}{(2\pi)^3}~\exp\left(-\frac{\omega_i}{T}\right)\right]~\exp\left(-\frac{U}{T}\right)~, \tag{1}$$

where V and T are the system volume and temperature, g is the number of internal degrees of freedom (degeneracy factor) of the particles, $\omega_i=(m^2+\mathbf{k}_i^2)^{1/2}$ is the dispersion relation of free particles with masses m. The particle interactions described by the function U in Eq. (1) are given by the sum over pair potentials:

$$U ~=~ \sum_{1\le i<j\le N} u_{ij}~. \tag{2}$$

In contrast to the usual statistical mechanic treatment of the pair potentials, the u_{ij} are assumed to be both coordinate and momentum dependent $u_{ij}\equiv u(\mathbf{r}_i,\mathbf{k}_i;~\mathbf{r}_j,\mathbf{k}_j)$. This generalization is necessary, if Lorentz contraction effects of hard spheres are to be taken into account. Introducing the Mayer functions

$$f_{ij} ~=~ \left[\exp\left(-\frac{u_{ij}}{T}\right)~-~1\right]~, \tag{3}$$

Eq. (1) can be presented as

$$Z_N(V,T) ~=~ \frac{1}{N!}\int d\mathbf{x}_1...d\mathbf{x}_N~\exp\left(-\frac{\omega_1+...+\omega_N}{T}\right)~\prod_{1\le i<j\le N}(1~+~f_{ij})~, \tag{4}$$

with the short notation $d\mathbf{x}_i\equiv g d\mathbf{r}_i d\mathbf{k}_i/(2\pi)^3$. Similarly to the standard procedure one can introduce the cluster integrals [5]

$$\begin{aligned}
b_1 ~&=~ \frac{1}{V}\int d\mathbf{x}_1~\exp\left(-\frac{\omega_1}{T}\right)~=~\frac{g~T^3}{2\pi^2}~K_2\left(\frac{m}{T}\right)~\equiv~\phi(T)~, & (5)\\
b_2 ~&=~ \frac{1}{2!V}\int d\mathbf{x}_1 d\mathbf{x}_2~\exp\left(-\frac{\omega_1+\omega_2}{T}\right)~f_{12}~, & (6)\\
b_3 ~&=~ \frac{1}{3!V}\int d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3~\exp\left(-\frac{\omega_1+\omega_2+\omega_3}{T}\right)~(f_{12}f_{13} & (7)\\
&+~ f_{12}f_{23}~+~f_{13}f_{23}~+~f_{12}f_{23}f_{13})~,\\
&...
\end{aligned}$$

and present the canonical partition function in the familiar form

$$Z_N(V,T) ~=~ \sum_{\{m_l\}}{}'\prod_{l=1}^{N}\frac{(Vb_l)^{m_l}}{m_l!}~, \tag{8}$$

where the summation in Eq. (8) is taken over all sets of non-negative integer numbers $\{m_l\}$ satisfying the condition

$$\sum_{l=1}^{N} l m_l ~=~ N ~. \tag{9}$$

Note, however, that the cluster integrals defined above are different from those used in standard statistical mechanics [5] as here nontrivial momentum integrations are included. Condition (9) makes the calculation of Z_N (8) rather complicated. This problem can be avoided in the grand canonical ensemble: the grand canonical partition function can be calculated explicitly ($z \equiv \exp(\mu/T)$):

$$\mathcal{Z}(V,T,\mu) ~\equiv~ \sum_{N=0}^{\infty} \exp\left(\frac{\mu N}{T}\right)~ Z_N(V,T) ~=~ \exp\left(~V~\sum_{l=1}^{\infty} b_l z^l~\right)~. \tag{10}$$

In the thermodynamical limit the pressure p and particle number density n are calculated in the grand canonical ensemble in terms of the asymptotic values of the cluster integrals:

$$p ~=~ T \lim_{V\rightarrow\infty} \frac{\ln \mathcal{Z}}{V} ~=~ T \sum_{l=1}^{\infty} b_l z^l ~, \tag{11}$$

$$n ~=~ \lim_{V\rightarrow\infty} \frac{1}{V}~\frac{\partial \ln \mathcal{Z}}{\partial z} ~=~ \sum_{l=1}^{\infty} l b_l z^l ~. \tag{12}$$

The virial expansion represents the pressure in terms of a series of particle number density and takes the form

$$p ~=~ T \sum_{l=1}^{\infty} a_l n^l ~. \tag{13}$$

Substituting p (11) and n (12) into Eq. (13) and equating the coefficients of each power of z, one finds the virial coefficients a_l in terms of the cluster integrals

$$a_1 ~=~ 1~, \quad a_2 ~=~ -~\frac{b_2}{b_1^2}~, \quad a_3 ~=~ \frac{4b_2^2}{b_1^4} ~-~ \frac{2b_3}{b_1^3}~, \quad ... \tag{14}$$

Let us recall, first, the derivation of the standard VdW excluded volume model. Then it is extended by adding the Lorentz contraction of the moving particles. Keeping the first two terms of the virial expansion (13) the following result is obtained:

$$p(T,n) ~=~ Tn~(1~+~a_2 n)~. \tag{15}$$

It is valid for small particle densities (i.e. $n << 1/a_2$). The usual (momentum independent) hard core potential for spherical particles with radius $r_{\rm o}$ is $u_{ij} = u(|\mathbf{r}_i - \mathbf{r}_j|)$. Here the function $u(r)$ equals to 0 for $r > 2r_{\rm o}$ and ∞ for $r < 2r_{\rm o}$. The second cluster integral (6) can easily be calculated in this case:

$$b_2 ~=~ -~\phi^2(T)~\frac{16\pi}{3} r_{\rm o}^3~. \tag{16}$$

Therefore $a_2 = 4\,v_{\rm o}$, where $v_{\rm o} = 4\pi r_{\rm o}^3/3$ is the particle hard core volume. The VdW excluded volume model is obtained as the extrapolation of Eq. (15) to large particle densities in the form

$$p(T,n) ~=~ \frac{Tn}{1~-~a_2 n}~. \tag{17}$$

For practical use the pressure is given as a function of T and μ independent variables, i.e. in the grand canonical ensemble. This is done by substituting $n = (\partial p/\partial\mu)_T$ into Eq. (17), which then turns into a partial differential equation for the function $p(T,\mu)$. For the VdW model (17) the solution of this partial differential equation can be presented in the form of a transcendental equation

$$p(T,\mu) \;=\; T\phi(T)e^{\mu/T}\exp\left(-\frac{a_2 p}{T}\right) \;\equiv\; p_{id}(T,\mu - a_2 p)\ . \tag{18}$$

Eq. (18) was first obtained in Ref. [6] using the Laplas transform technique. With $p(T,\mu)$ (the solution of Eq. (18)) the particle number density, entropy density and energy density are calculated as ($\nu = \mu - a_2\, p(T,\mu),\ a_2 = 4\, v_o$):

$$n(T,\mu) \;\equiv\; \left(\frac{\partial p(T,\mu)}{\partial\mu}\right)_T \;=\; \frac{n_{id}(T,\nu)}{1\;+\;a_2 n_{id}(T,\nu)}\ , \tag{19}$$

$$s(T,\mu) \;\equiv\; \left(\frac{\partial p(T,\mu)}{\partial T}\right)_\mu \;=\; \frac{s_{id}(T,\nu)}{1\;+\;a_2 n_{id}(T,\nu)}\ , \tag{20}$$

$$\epsilon(T,\mu) \;\equiv\; Ts\;-\;p\;+\;\mu n \;=\; \frac{\epsilon_{id}(T,\nu)}{1\;+\;a_2 n_{id}(T,\nu)}\ . \tag{21}$$

Here the superscripts *id* in the thermodynamical functions (18–21) indicate those of the ideal gas.

The excluded volume effect accounts for the blocked volume of two spheres when they touch each other. If hard-sphere particles move with relativistic velocities it is necessary to include their Lorentz contraction in the rest frame of the fluid. The model suggested in Ref. [4] is not satisfactory: the parameter $a_2 = 4\, v_o$ of the VdW excluded volume model is confused there with the proper volume of an individual particle – the contraction effect is introduced for the proper volume of each particle. In order to get the correct result it is necessary to account for the excluded volume of two Lorentz contracted spheres.

Let $\mathbf{r}_i$ and $\mathbf{r}_j$ be the coordinates of the i-th and j-th particle, respectively, and $\mathbf{k}_i$ and $\mathbf{k}_j$ be their momenta, $\hat{\mathbf{r}}_{ij}$ denotes the unit vector $\hat{\mathbf{r}}_{ij} = \mathbf{r}_{ij}/|\mathbf{r}_{ij}|$ ($\mathbf{r}_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$). Then for a given set of vectors $(\hat{\mathbf{r}}_{ij}, \mathbf{k}_i, \mathbf{k}_j)$ for the Lorentz contracted rigid spheres of radius r_o there exists the minimum distance between their centers $r_{ij}(\hat{\mathbf{r}}_{ij}; \mathbf{k}_i, \mathbf{k}_j) = \min|\mathbf{r}_{ij}|$. The dependence of the potentials u_{ij} on the coordinates $\mathbf{r}_i, \mathbf{r}_j$ and momenta $\mathbf{k}_i, \mathbf{k}_j$ can be given in terms of the minimal distance as follows

$$u(\mathbf{r}_i, \mathbf{k}_i; \mathbf{r}_j, \mathbf{k}_j) = \begin{cases} 0\,, & |\mathbf{r}_i - \mathbf{r}_j| > r_{ij}\left(\hat{\mathbf{r}}_{ij}; \mathbf{k}_i, \mathbf{k}_j\right)\ , \\ \infty\,, & |\mathbf{r}_i - \mathbf{r}_j| \le r_{ij}\left(\hat{\mathbf{r}}_{ij}; \mathbf{k}_i, \mathbf{k}_j\right)\ . \end{cases} \tag{22}$$

The general approach to the cluster- and virial expansions described above is valid for this momentum dependent potential, and it leads to Eqs. (17,18) with

$$\begin{aligned} a_2(T) \;&=\; \frac{1}{2\phi^2(T)}\int\frac{d\mathbf{k}_1 d\mathbf{k}_2}{(2\pi)^6}\ \exp\left(-\frac{\omega_1\;+\;\omega_2}{T}\right)\ \times \\ &\times\ \int d\mathbf{r}_{12}\ \Theta\left(r_{12}(\hat{\mathbf{r}}_{12}; \mathbf{k}_1, \mathbf{k}_2)\;-\;|\mathbf{r}_{12}|\right)\ . \end{aligned} \tag{23}$$

The new feature is the temperature dependence of the excluded volume a_2 (23) which is due to the Lorentz contraction of the rigid spheres. The pressure and particle number density are still given by Eqs. (18,19), but with temperature dependent $a_2(T)$ (23). However, Eqs. (20, 21) are now modified, e.g.

$$\epsilon(T,\mu) \;=\; \frac{\epsilon_{id}(T,\nu)\;-\;p^2\;da_2(T)/dT}{1\;+\;a_2 n_{id}(T,\nu)}\;. \tag{24}$$

In contrast to Eq. (21) the energy density (24) contains the extra term which appears also in the entropy density. The excluded volume $a_2(T)$ (23) is always smaller than $4\,v_{\rm o}$. It has been proven rigorously that $a_2(T)$ is a monotonously decreasing function of T and, therefore, the additional term in Eq. (24) is always positive. Let us introduce the notation

$$a_2(T) \;=\; 4\,v_{\rm o}\;f(T)\;. \tag{25}$$

The function $f(T)$ depends on the T/m ratio. It can be calculated numerically and its behavior is shown in Fig. 1. The simple analytical formula

$$f(T) \;=\; c\;+\;(1-c)\;\frac{\rho_s(T)}{\phi(T)} \tag{26}$$

with

$$c \;=\; \left(1\;+\;\frac{74}{9\pi}\right)^{-1}\;, \qquad \rho_s \;=\; \frac{g}{(2\pi)^3}\int d\mathbf{k}\;\frac{m}{\omega}\;exp\left(-\frac{\omega}{T}\right)\;,$$

is found to be valid with an accuracy of a few percents for all temperatures. The asymptotic behavior of $f(T)$ is the following: $1-{\rm O}(T/m)$ at $T<<m$ and $c+{\rm O}(m/T)$ at $T>>m$.

If one assumes that all types of hadrons have at rest the same hard core radius then the Lorentz contraction effect leads to different VdW excluded volumes for moving particles with different masses: for light particles (e.g. pions) the excluded volume (at given T) is smaller than that for heavy ones. Fig. 1 shows that at $T\cong 150$ MeV the value of a_2 in the nucleon gas ($m\cong 939$ MeV) decreases by 10% in comparison to its nonrelativistic value $4\,v_{\rm o}$, whereas for pions ($m\cong 140$ MeV) a_2 shrinks at the same T by almost a factor 2. This is simply because light particles are more relativistic than heavy ones at given temperature typical for high energy nuclear collisions, $T=120\div 170$ MeV.

As an example, Fig. 2 shows the particle number density of the pion gas ($\mu=0, g=3$) with $r_{\rm o}=0.5$ fm. The particle number density is calculated according to Eq. (19) for three different models: the ideal pion gas ($a_2=0$), the VdW model with constant excluded volume ($a_2=4v_{\rm o}$) and the VdW model with Lorentz contraction ($a_2(T)$ is given by Eq. (23)). It can be seen from Fig. 2 that at low T the pion density is small and excluded volume corrections are unimportant. Therefore, all three models are similar. The situation changes with increasing T: the suppression due to the excluded volume effects are large and different for $a_2=4v_{\rm o}$ and $a_2(T)$ (23). The ratios of particle number densities and energy densities of the pion gas for two versions of the VdW model ($a_2=4v_{\rm o}$ and $a_2(T)$ (23)) are shown in Fig. 3 as functions of the temperature. From Fig. 3 one can observe the deviations between these two models. These deviations increase with temperature. They are larger for the energy density due to the additional positive term in Eq. (24).

In conclusion, the traditional cluster and virial expansions can be consistently generalized to momentum dependent pair potentials. Hard-core potentials with Lorentz contraction effects lead to a VdW model with a temperature dependent excluded volume $a_2(T)$ (23). For light particles the effect of Lorentz contraction is, evidently, stronger than for heavy ones. Note that smaller values of the pion hard core radius r_π were introduced in Refs. [7, 8] within the standard VdW excluded volume model to fit hadron yield data better. The smaller value of the pion excluded volume appears as a consequence of stronger Lorentz contraction for light particles.

Acknowledgments

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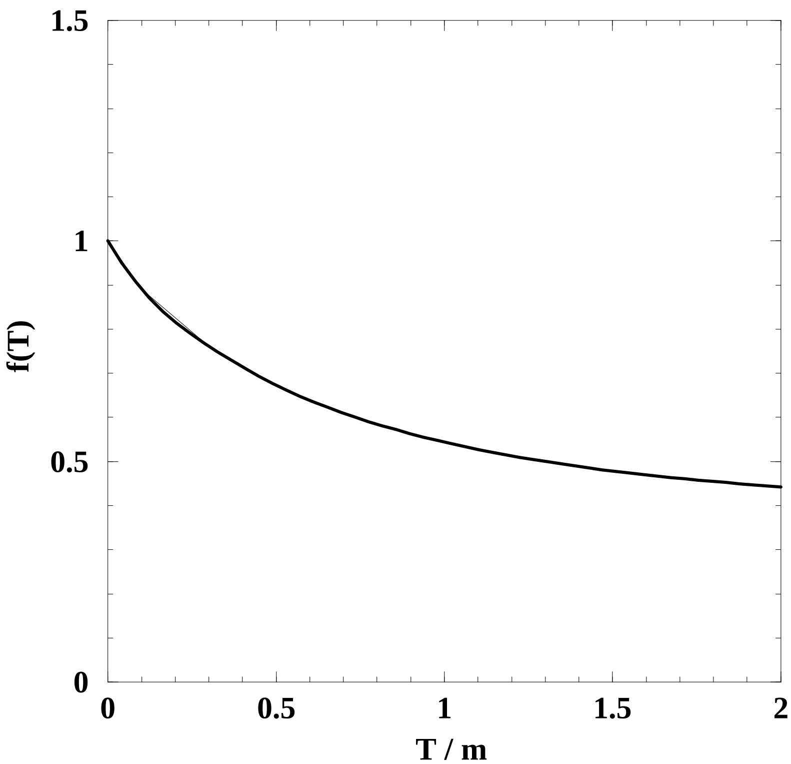

Fig. 1. $f(T)$ as the function of the temperature-to-mass ratio. For heavy particles (e.g., nucleons $m >> T$) the volume reduction is just a few per cents, whereas for pions ($m \approx T$) it is about 50%.

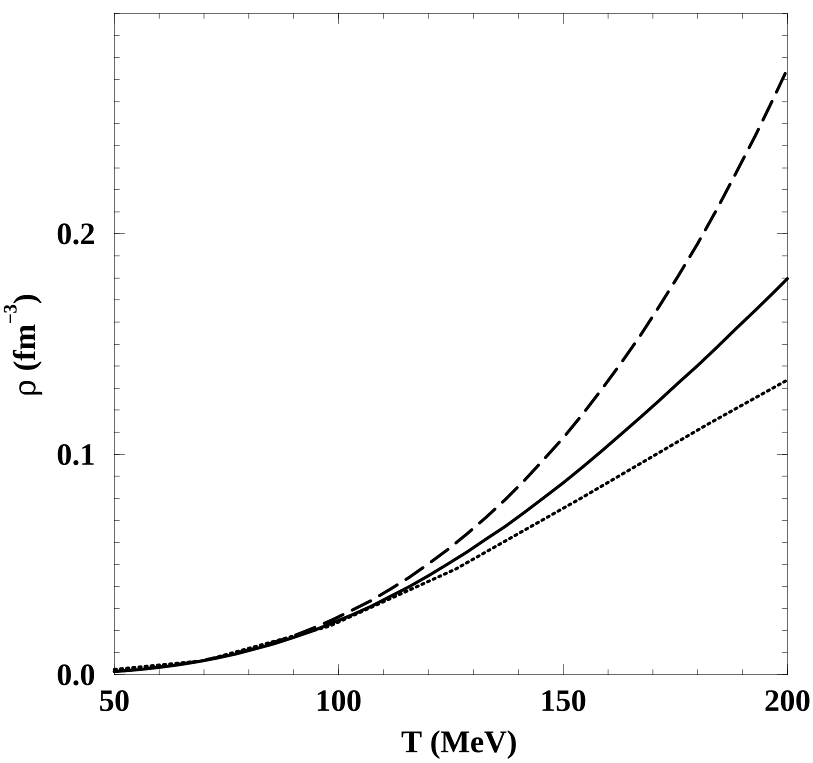

Fig. 2. The particle number density for three models of the pion gas ($\mu = 0, g = 3$): the solid line corresponds to the VdW model of the Lorentz contracted spheres ($r_{\rm o} = 0.5$ fm), the dashed one corresponds to the ideal gas of point-like particles, and the dotted one corresponds to the VdW model without Lorentz contraction for the spheres of a constant radius 0.5 fm.

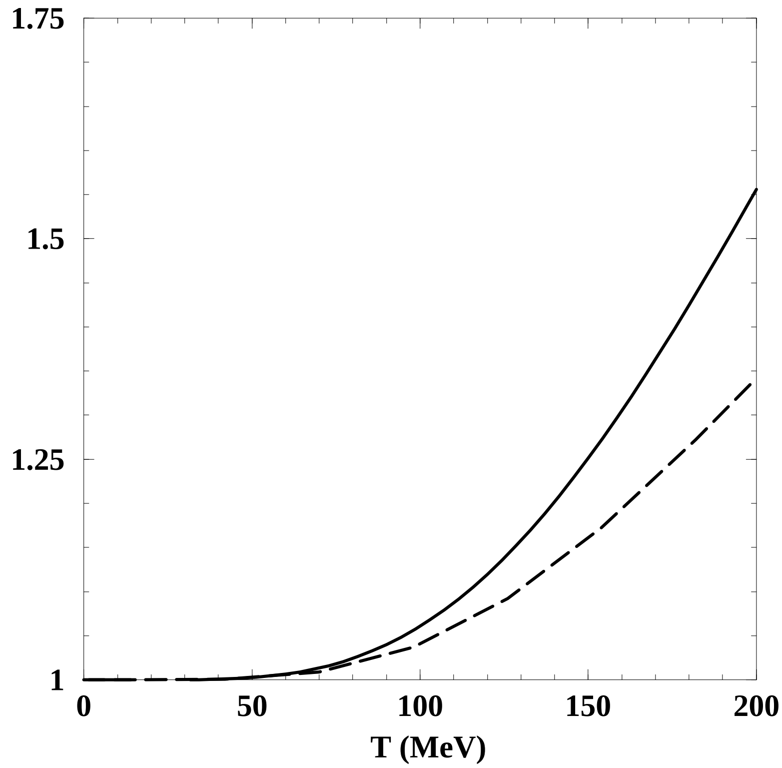

Fig. 3. The dashed line shows the ratio of the particle number densities of the pion gas ($g = 3, \mu = 0, r_o = 0.5$ fm): the VdW model with Lorentz contraction divided by the VdW model without Lorentz contraction. The solid line shows a similar ratio for the energy densities.